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Phase transitions on fractals: I. Quasi-linear lattices

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Abstract. Magnetic spin models and resistor networks are studied on certain self-similar fractal lattices, which are described as 'quasi-linear', because they share a significant property of the line: finite portions can be isolated from the rest by removal of two points (sites). In all cases, there is no long-range order at finite temperature. The transition at zero temperature has a discontinuity in the magnetisation, and the associated magnetic exponent is equal to the fractal dimensionality, D . When the lattice reduces to a non-branching curve the thermal exponent $\nu^{-1} = \gamma$ is equal to D . When the lattice is a branching curve, γ is related, respectively, to the dimensionality of the single-channel segments of the curve (for the Ising model), or to the exponent describing the resistivity (for models with continuous spin symmetry).

1. Introduction

In recent years, there has been much interest in physical systems with non-integer dimensionalities, coming from two directions. The theory of critical phenomena has made much progress using ϵ expansions (Wilson and Fisher 1972, Brézin and Zinn-Justin 1976, Wallace and Zia 1979), which involve formal analytic continuations of expressions concerning the Euclidean dimensionality d . These calculations implicitly involve an underlying fractional-dimensional space that is always assumed to be translationally invariant, although no such space has ever been implemented. An important result of these and other studies on translationally invariant systems is the concept of universality, which implies that, given the symmetry of the order parameter and the range of interactions, a system's critical properties depend solely on the dimensionality d (see e.g. Fisher 1974, Aharony 1976). Unfortunately, the purely formal character of the analytic continuation has prevented an actual test of universality for non-integer dimensionalities.

By contrast, fractals (Mandelbrot 1977, 1982) are fully explicitly described geometric shapes of fractional dimensionality. They can be scale invariant, but cannot be translationally invariant. Many of the self-similar fractals are constructed by recursive replacement of segments, triangles, squares, etc by more complex shapes called 'generators', which are made of smaller segments, triangles, squares, etc. At present, fractals are the only explicit geometric shapes, with non-integer dimensionalities, which allow the results derived by abstract analytic continuations to be tested. Let us also point out that Gefen *et al* (1983) show that a formal translationally invariant fractional-dimensional space can be implemented arbitrarily closely by suitable low-lacunarity fractal lattices. Furthermore, fractals provide useful models for real physical

systems, e.g. polymer chains (Mandelbrot 1977, 1982, Havlin and Ben Avraham 1982) or percolating clusters (Stanley *et al* 1976, Mandelbrot 1977, 1978, 1982, Stanley 1977, Kirkpatrick 1979, Stauffer 1979, Gefen *et al* 1981).

In Gefen *et al* (1980), we reported on a preliminary study of critical phenomena on fractal lattices. We found that these phenomena depend on the fractal dimensionality, D , but in addition depend on many topological characteristics of the lattices including the topological dimensionality D_T , the orders of ramification R , the connectivity Q and the lacunarity L . The present paper is the first in a series of three, which describe in detail our systematic study of the behaviour of spin models and resistor networks on various fractal lattices. This paper is devoted to Koch curves that we call quasi-linear because they share a significant property of one-dimensional chains; finite portions can be isolated from the rest by removal of two points (sites). Hence the minimum order of ramification (excluding endpoints) is 2. Paper II will describe the behaviour of systems sited on Sierpiński gaskets, which were proposed as models for the backbone of the infinite cluster at percolation (Gefen *et al* 1981, Mandelbrot 1982). The gasket's orders of ramification are greater than 2 but finite. One finds no long-range magnetic order at finite temperature, and one may solve the models exactly. Paper III will contain the analysis on Sierpiński carpets. In that case R is infinite, there is a finite transition temperature, and the calculations are approximate.

Several earlier papers also considered critical phenomena on specific geometrical systems (e.g. Nelson and Fisher 1975, Dhar 1977, 1978). The present series of papers generalises these discussions, with emphasis on the variation of critical properties with the various geometrical characteristics.

The outline of this paper is as follows. Section 2 describes various quasi-linear Koch curves and explains the relevant geometric facts. Section 3 contains an analysis of physical systems on quasi-linear Koch curves that are non-branching. This includes discrete-symmetry models (e.g. Ising models), models with continuous spin symmetry ($n > 2$) and resistor networks. Scaling and hyperscaling relations for these models are also discussed. Section 4 presents an analogous analysis for quasi-linear Koch curves that are branching. Section 5 is a short summary.

2. Geometry

Many fractals are constructed starting with some given shape called the 'initiator' (it may be a segment, square, triangle, etc). One then replaces the initiator by suitably rescaled copies of a shape called the 'generator'. This procedure is continued down to a 'microscopic' length scale, at which it is stopped. The resulting shape is self-similar on all the intermediate length scales. In the fractal lattices considered in the present series of papers, the generator is identical to the initiator. The basic spins (or resistors) are placed on the sites (or bonds) of the smallest scale lattice.

Some examples of non-branching Koch curves are shown in figure 1. To achieve a self-similar structure we recursively replace each segment by N smaller segments of length $1/b$. Then the fractal dimensionality D is defined by $b^D = N$, or

$$D = \ln N / \ln b. \quad (2.1)$$

Every fractal embedded in d -dimensional Euclidean space satisfies $D \leq d$. Therefore, if the curve of figure 1(*f*) is to be self-avoiding, it must be embedded in a Euclidean

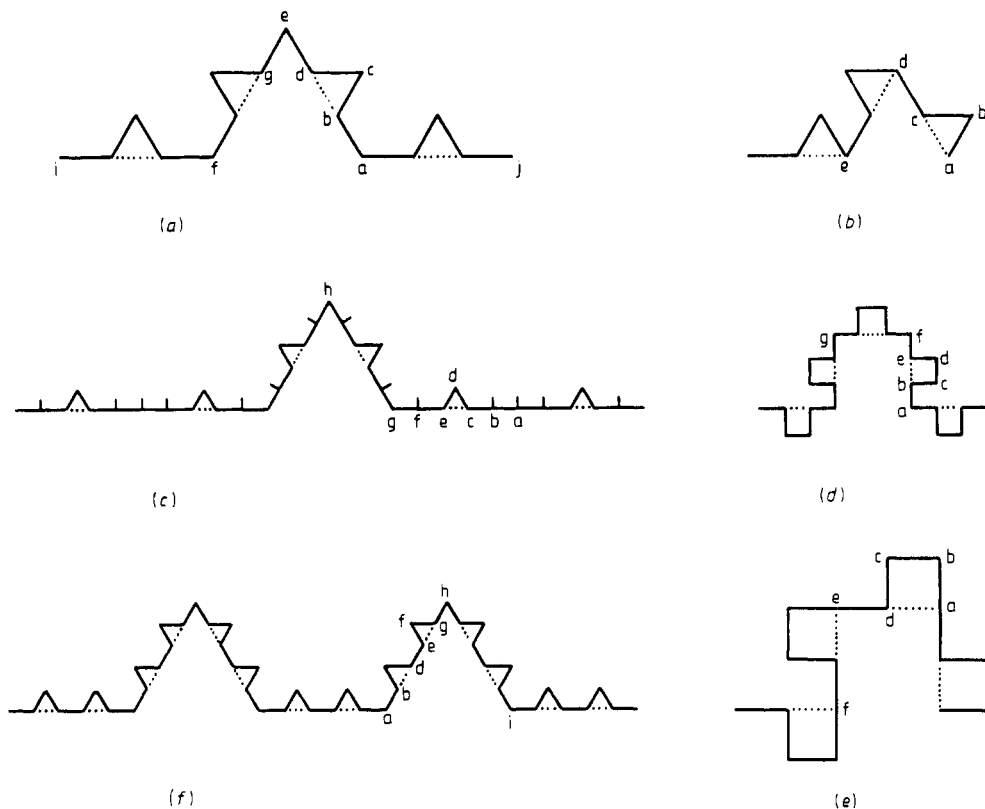


Figure 1. Two construction stages of non-branching Koch curves. Dotted lines denote spin interactions that are not part of the self-similar structure. (a) $D = \ln 4 / \ln 3$, (b) $D = \ln 3 / \ln 2$, (c) $D = \ln 6 / \ln 5$, (d) $D = \ln 5 / \ln 3$, (e) $D = \ln 7 / \ln 5$, (f) $D = \ln 4 / \ln 2 = 2$.

space with $d > 2$. Note that, instead of counting the lattice segments, it is more natural in the case of spin systems to count the number of lattice sites, the end sites being counted for $\frac{1}{2}$. However, these numbers coincide, and hence yield the same value of D .

The topological dimensionality D_T (Mandelbrot 1977, 1982) is defined recursively as $D_T = D'_T + 1$, where D'_T is the topological dimensionality of the 'cutting set', that is, of the set which cuts the lattice into two separate pieces; D'_T can be found recursively in the same way. All the lattices examined in this paper, however ramified (see below), can be cut at a finite number of points, hence $D'_T = 0$ and $D_T = 1$. This value is also characteristic of the standard curves (lines, circles, ...), hence our lattices are curves from the topological viewpoint.

An important parameter in our discussion is the order of ramification R . At a point P , R measures the smallest number of significant interactions which one must cut in order to isolate an arbitrary bounded group of points on the curve, surrounding P (Menger 1932, Mandelbrot 1982). The maximum and minimum values of R obey

$$R_{\max} \geq 2R_{\min} - 2. \tag{2.2}$$

Many curves include a few points of anomalously low order of ramification, such as endpoints where $R = 1$. In such cases, equation (2.2) is verified trivially. However,

it also holds when the exceptional points are neglected. When $R_{\max} = R_{\min}$, the curve is to be called homogeneous. This requires $R = 2$ or $R = \infty$. Lattices with $R = 2$ are to be called non-branching. When $R_{\max} = 2R_{\min} - 2$ the curve is to be called quasi-homogeneous.

Examples of branching Koch curves are shown in figure 2. These curves are quasi-linear but inhomogeneous, in the sense that the order of ramification takes the value $R = 2$ at some but not all points. At the 'branching' points one has $R > 2$.

Curves with $D \geq 2$ cannot be drawn in the plane without extensive self-overlap, but in many cases can and should be embedded without self-overlap in Euclidean spaces with $d > D \geq 2$.

For inhomogeneous curves, yet another dimensionality can be defined formally. While $D = \log B(k) / \log b^k$ where $B(k)$ is the number of segments ('bonds') at the k th iteration, one can define \tilde{D} as $\tilde{D} = \log S(k) / \log b^k$, where $S(k)$ is the number of

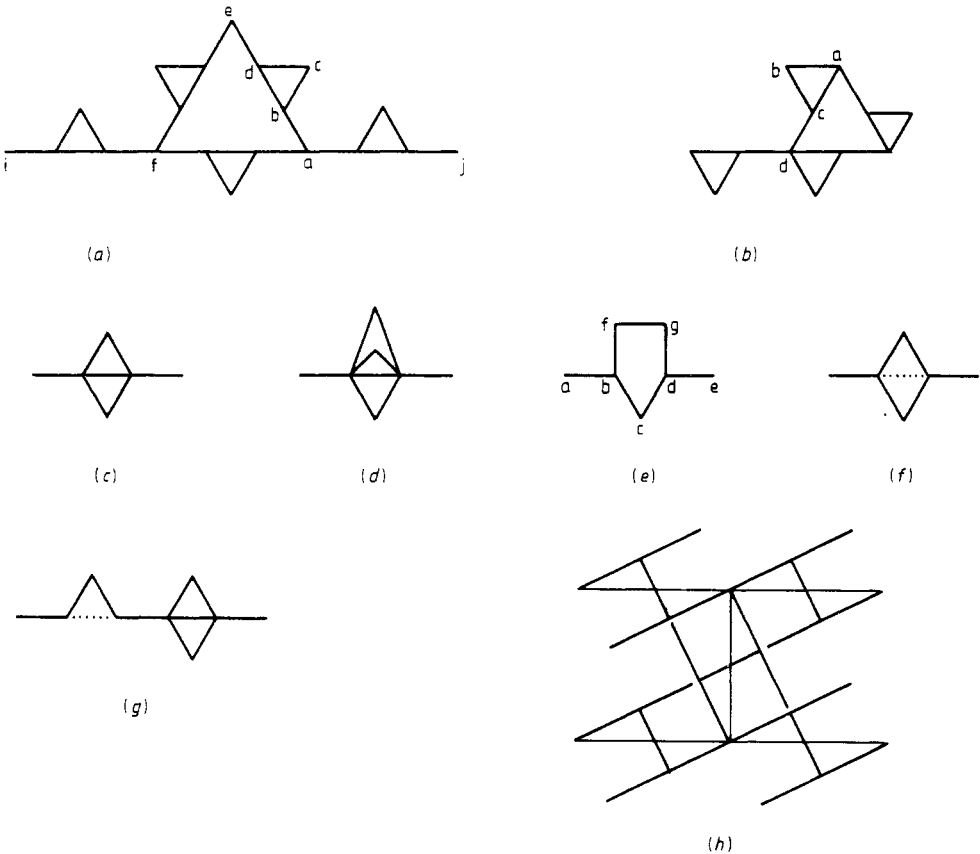


Figure 2. Branching Koch curves. Dotted lines denote spin interactions that are not part of the self-similar structure. In (c), (d), (e), (f), (g) only one construction stage is shown. In (h), the thin lines denote the first decoration step. Dots denote endpoints, which are infinitely close to nodes of the tree, but do not actually touch them. We assume that physical inter-spin interactions proceed only along the links of the tree. The fractal dimensionalities are (a) $D = \ln 5 / \ln 3$, (b) $D = \ln 4 / \ln 2 = 2$, (c) $D = \ln 7 / \ln 3$, (d) $D = \ln 9 / \ln 3 = 2$, (e) $D = \ln 7 / \ln 3$, (f) $D = \ln 6 / \ln 3$, (g) $D = \ln 10 / \ln 5$, (h) $D = \ln 5 / \ln 5^{1/2} = 2$.

sites at the k th iteration (spin degrees of freedom). Fortunately, \tilde{D} is not a separate notion, because after many rescaling iterations \tilde{D} approaches D . On the lattice of figure 2(a), this assertion is obvious. Indeed, each point has a coordination number 2 or 3, hence $B(k) < S(k) < 3B(k)/2$, which implies $\tilde{D} = D$. A less obvious situation arises for figure 2(b), where (in the limit) there exist points with an infinite order of ramification. However, $D = 2$, and it is easy to evaluate \tilde{D} by observing that

$$S(k+1) = S(k) + 2B(k), \quad (2.3)$$

which becomes

$$S(k+1) = S(k) + 2 \times 4^k. \quad (2.4)$$

To solve (2.4), the standard method is to form the generating function, defined as

$$g(x) \equiv \sum_{k=1}^{\infty} S(k)x^k. \quad (2.5)$$

From equations (2.4) and (2.5) we obtain

$$[g(x) - S(1)x]/x = g(x) + 8x/(1-4x). \quad (2.6)$$

Substituting $S(1) = 4$, and manipulating the algebra, we obtain

$$g(x) = 4[(1-2x)/(1-4x)]x/(1-x). \quad (2.7)$$

Hence

$$S(k) = \frac{2}{3}4^k(1+2/4^k), \quad (2.8)$$

and $\tilde{D} = 2 = D$. The above examples suggest that both branching and non-branching Koch curves with any $D > 1$ can be constructed.

3. Non-branching quasi-linear Koch curves

In this section we analyse some physical models placed on non-branching quasi-linear Koch curves. In the case of spin models, the spins are put on the lattice sites, and are assumed to interact with their nearest neighbours on the 'microscopic' scale. In the case of resistor networks, the lattice bonds are assumed to possess a microscopic resistance r . We start by describing a detailed calculation on the lattice of figure 1(a), and then generalise.

As noted above, the geometrical structure in figure 1(a) has the fractal dimensionality $D = \ln 4 / \ln 3$, because the length scale is changed by a factor $b = 3$, and the number of new segments is $N = 4$. In a preliminary step, we restrict the interactions (or resistors) to nearest neighbours along the curve, that is, to the segments $\langle ab \rangle$, $\langle bc \rangle$, $\langle cd \rangle$ and $\langle de \rangle$, which form a one-dimensional chain. The only effect of self-similar wiggleness is that the number of bonds between two points which are a distance x apart along a straight line is not equal to x but to x^D in dimensionless units. Hence, one can apply all the known one-dimensional results, with x replaced by x^D . Since all nearest-neighbour spin-spin correlations decay exponentially, on the line, we now expect them to decay as $\exp(-x^D/\xi_1)$, where ξ_1 is the one-dimensional correlation length. This result may be rewritten as

$$\xi_D = \xi_1^{1/D}. \quad (3.1)$$

For the Ising model, $\xi_1 \propto e^{2K}$, where $K = J/kT$ and J is the nearest-neighbour exchange. Using $t = e^{-2K}$ as the low temperature scaling field, equation (3.1) may be written as

$$\xi_D \sim t^{-\nu} \quad \text{with } \nu = 1/D. \tag{3.2}$$

For continuous spin models, $\xi_1 \propto K = t^{-1}$, and again $\nu = 1/D$. If the bonds represent resistors, the resistance scales as x^D . Analogous results are expected for all the structures in figure 1.

The above results are perfectly straightforward. Next, in order to make the problem less trivial, we add a microscopic interaction between sites that are not nearest neighbours along the curve, but are nearest neighbours in the plane, such as b and d, but not a and f, or d and g. We now formulate renormalisation group recursion relations for the Ising model, the n -component spin model and resistor networks.

3.1. Ising model

Our basic microscopic Hamiltonian has exchange interactions JS_iS_j , with $S_i = \pm 1$, between $\langle ab \rangle$, $\langle bc \rangle$, $\langle cd \rangle$, $\langle de \rangle$ and $\langle bd \rangle$. In the absence of magnetic field, the trace over S_b , S_c and S_d yields an effective interaction between a and e,

$$\exp(\tilde{K}S_aS_e + A) = \text{Tr}_{S_b, S_c, S_d} \exp[K(S_aS_b + S_bS_c + S_cS_d + S_dS_e + S_bS_d)], \tag{3.3}$$

with

$$\tilde{\tau} = \tanh \tilde{K} = \tau^3(1 + \tau)/(1 + \tau^3), \tag{3.4}$$

where $\tau \equiv \tanh K$. To define a renormalisation group scheme, a renormalised coupling K' corresponding to a larger scale must be defined in such a way that the partition function is invariant under this transformation. The renormalised coupling is between spins $\langle aj \rangle$, $\langle ae \rangle$, $\langle ef \rangle$, $\langle fi \rangle$ and also $\langle af \rangle$. Defining

$$X(\tau) = \tau^3(1 + \tau)/(1 + \tau^3), \tag{3.5}$$

it is easy to check that $\tau' = \tanh K'$ is given by

$$X(\tau') = [X(\tau)]^4. \tag{3.6}$$

The only fixed points are at $T = 0$ ($K = \infty$) and $T = \infty$ ($K = 0$). Linearising near $T = 0$, and writing $t' = b^y t$ with $t = \exp(-2K)$, yields

$$\nu^{-1} = y = \ln 4 / \ln 3 = D. \tag{3.7}$$

Thus we have shown that the additional coupling between b and d does not change the simple result (3.2).

We now introduce a small magnetic field. To find the magnetic exponent we take $\tau, \tau' \rightarrow 1$ (zero temperature). The only contributions to the trace over S_b, S_c and S_d will now come from $S_b = S_c = S_d = S_a = S_e$. Thus

$$\begin{aligned} \exp(h'S_a) &= \exp[\frac{1}{2}h'(S_a + S_e)] = \text{Tr}_{S_b, S_c, S_d} \exp[\frac{1}{2}h(S_a + S_e) + h(S_b + S_c + S_d)] \\ &= \exp(4hS_a), \end{aligned} \tag{3.8}$$

i.e.

$$h' = 4h \tag{3.9}$$

(h is measured in units of $k_B T$). There is a factor of $\frac{1}{2}$ in front of hS_a because the field on each spin is divided between the bonds which connect this spin to its neighbours. Writing $h' = b^x h$, the magnetic exponent is found to be

$$x = \ln 4 / \ln 3 = D. \quad (3.10)$$

The free energy per spin (in units of $k_B T$), F , is expected to scale as b^{-D} , since the number of spins is rescaled by $N = b^D$. Ignoring the sum over the constants (A in equation (3.3)), we expect the general scaling relation

$$F(t, h) = b^{-D} F(b^D t, b^D h). \quad (3.11)$$

Eliminating b yields

$$F(t, h) = t f(h/t), \quad (3.12)$$

and we can now identify all the thermodynamic exponents, e.g.

$$\alpha = 1, \quad \beta = 0, \quad \delta = \infty. \quad (3.13)$$

Note that these results coincide with the results on the line (e.g. Nelson and Fisher 1975), as might be anticipated.

The transition at $T = 0$ is of first order, since the magnetisation 'jumps' from zero to unity. It is interesting to note that the fixed point at $T = 0$ has a magnetic eigenvalue equal to b^D . The fractal dimensionality D thus replaces the usual dimensionality d , expected to appear for a 'discontinuity fixed point' (Nienhuis and Nauenberg 1975, Berker and Fisher 1982). Except for the replacement of d by D , we recover all the usual modifications of hyperscaling relations at zero-temperature transitions (Baker and Bonner 1975).

3.2. Models with $n > 1$

We now replace the Ising interaction by $J(S_i \cdot S_j)$, where S_i is an n -component unit vector. Denoting by $\lambda_n(K)$ the (normalised) nearest-neighbour spin correlation function in a linear chain (Stanley 1969), the correlation function between a and e in figure 1(a) is easily found to be

$$X(K) = \lambda^2(K) \lambda(K + K_1), \quad (3.14)$$

where $\lambda(K_1) = \lambda^2(K)$. The new coupling constant K' is now found through

$$X(K') = X(K)^4. \quad (3.15)$$

Note that equations (3.14) and (3.15) also contain equations (3.5) and (3.6), since for the Ising case $\lambda(K) = \tanh K$. They easily generalise to any other problem, in which the one-dimensional nearest-neighbour correlation function (i.e. the ratio of the two largest eigenvalues of the transfer matrix) is known.

When the temperature is very low and $n > 1$, one has $\lambda_n(K) \approx 1 - a/K$, with $a = (n - 1)/2$ (Stanley 1969). Thus, $\lambda(K_1) \approx 1 - 2a/K$, i.e. $K_1 \approx K/2$, and $X \approx 1 - 8a/3K$. Finally, equation (3.14) yields $K' = K/4$. Using $t = 1/K$ and $t' = b^y t$, we thus again identify $v^{-1} = y = \ln 4 / \ln 3 = D$. One similarly recovers $x = D$, and all the other results in equation (3.13).

3.3. Resistors

The bonds in figure 1(a) (including one between b and d) now represent resistors. The net resistance between a and e is $2r + (1/r + 1/2r)^{-1} = 8r/3$. The new basic resistance r' is now found via $8r'/3 = 4(8r/3)$, i.e. $r' = 4r$. Writing $r' = b^{\tilde{\zeta}}r$, we identify

$$\tilde{\zeta} = \ln 4 / \ln 3 = D. \tag{3.16}$$

Again, this agrees with the simple quasi-linear scaling presented in the beginning of this section.

3.4. Generalisations

The procedure used above consisted of two basic steps. We first traced over the sites b, c and d, and obtained an effective bond (related to $X(K)$) between a and e. At this stage, the range between i and j (figure 1(a)) simply contained $b^D = 4$ one-dimensional bonds. We next introduced the bond between a and f, and identified the new coupling K' via equation (3.15). It is now easy to convince oneself that the generalisation to any other non-branching Koch curve is

$$X(K') = X(K)^N, \tag{3.17}$$

where $N = b^D$. The function $X(K)$ is expected to be analytic in the small variable t (equal to e^{-2K} for $n = 1$, or to K^{-1} for $n > 1$). Writing $X(K) \approx 1 - at$, equation (3.17) immediately yields

$$t' = Nt = b^D t \tag{3.18}$$

and $y = D$.

It is similarly easy to see that $h' = Nh$, since each spin replaces N spins on the lower scale. Thus, $x = D$. The result $\tilde{\zeta} = D$ follows in exactly the same manner.

We conclude that for all non-branching Koch curves one has $y = x = \tilde{\zeta} = D$, as expected from their quasi-linear nature.

4. Branching quasi-linear Koch curves

We now turn to branching Koch curves, such as those shown in figure 2. Figure 2(a) differs from figure 1(a) in that the bond (bd) is now part of the self-similar structure. Thus, the new coupling constant K' is simply given by

$$\lambda(K') = X(K), \tag{4.1}$$

with $X(K)$ given in equation (3.14).

4.1. Ising model

For the Ising case, $\lambda(K) = \tanh K$, so that

$$\tanh K' = \tanh^2 K \tanh(K + K_1), \tag{4.2}$$

with $\tanh K_1 = \tanh^2 K$. At large K , $\tanh K \approx 1 - 2t$, with $t = e^{-2K}$, and to leading order $t' = 2t$, i.e. $y = \ln 2 / \ln 3$. This result is easily generalisable. Only one-channel links will yield factors of $\tanh K \approx 1 - 2e^{-2K}$ on the right-hand side of (4.2). Bonds which

couple in parallel yield factors like $\tanh(K + K_1) \approx 1 - 2e^{-2(K+K_1)} = 1 + o(e^{-2K})$. To leading order in e^{-2K} , all the latter factors are equal to unity, implying that the corresponding spins are practically parallel to each other. Denoting by N' the number of one-channel links, we thus conclude that

$$\nu^{-1} = y = \ln N' / \ln b, \quad n = 1. \quad (4.3)$$

Clearly, $y < D = \ln N / \ln b$.

We now add a small magnetic field, at zero temperature. Since the branching Koch curves are inhomogeneous, different sites have different coordination numbers. One iteration of the renormalisation group will thus generate different magnetic fields at different sites. Denoting the magnetic field at a site with q neighbours by h_q , we shall now have a set of linear recursion relations for the h_q 's. In the example of figure 2(a) we must consider h_2 (at sites c, e) and h_3 (at sites a, b, d). At zero temperature, all the spins are aligned parallel to each other. Dividing the contribution from each new bond equally between the spins at its ends, we find

$$h'_2 = h_2 + (h_2 + 2h_3), \quad h'_3 = h_3 + \frac{3}{2}(h_2 + 2h_3). \quad (4.4)$$

The eigenvalues of the corresponding 2×2 matrix are easily found to be 5 and 1. The former is equal to b^D , as expected by the general argument near a discontinuity fixed point. The latter signals the existence of a 'marginal' variable. After many iterations, the fields h_2 and h_3 will be proportional to their values for the eigenvector of the larger eigenvalue, which we find to be $h_2/h_3 \rightarrow \frac{2}{3}$.

We followed the same procedure for all the branched Koch curves in figure 2, and we always find one eigenvalue b^D , whose eigenvector obeys $h_q = aq$. All the other eigenvalues are found to be equal to unity. In general, denoting by \tilde{q}_A the number of generators adjacent to the point A, we have a recursion relation of the form

$$h'_{\tilde{q}_A} = h_{q_A} + \frac{1}{2} \tilde{q}_A \sum_{q'} m_q h_{q'}, \quad (4.5)$$

where m_q is the number of times that a site with q neighbours appears as an internal point on the basic generator. We now write down a similar equation for a point B, which is an endpoint of a generator connected to A. Adding these two equations results in

$$h'_{\tilde{q}_A} + h'_{\tilde{q}_B} = h_{q_A} + h_{q_B} + \frac{1}{2} (\tilde{q}_A + \tilde{q}_B) \sum_{\tilde{q}} m_q h_{q'}. \quad (4.6)$$

The eigenvalue $N = b^D$, with eigenvector $h_q = aq$, results from the identity

$$\sum_q m_q q = 2N - 2(q_A + q_B) / (q'_A + q'_B). \quad (4.7)$$

Notice that in a tree (e.g. figure 2(h)) there is a single one-channel path between any two points, dressed with dangling ends. Hence, $y_1 = y_{n>1} < D$. (In the example of figure 2(h) $y_{n=1} = y_{n>1} = \log 3 / \log \sqrt{5} \approx 1.365$.) The corresponding magnetic exponents are calculated as above. The general rule of having a magnetic eigenvalue b^D at a discontinuity fixed point is thus obeyed, with the fractal dimensionality D .

4.2. Models with $n > 1$ and resistors

In the continuous models it is no longer true that spins connected via more than one channel are practically parallel to each other. Now one has $\lambda_n(K) \approx 1 - a/K$, so that

for figure 2(a) we have $\lambda(K') = X(K) \approx 1 - 8a/3K$, or $K' = 3/8 K$, i.e. $t' = 8/3t$. We thus find $\nu^{-1} = y = \ln(8/3)/\ln 3$.

To generalise this result, it is useful to note the analogy between continuous spin models at low temperature and resistor networks. The derivation of equation (3.13) was based on the fact that the net coupling constant of two bonds which connect the same two sites (in parallel) is the sum of the corresponding bonds, $(K + K_1)S_i S_j$. On the other hand, the effective coupling constant resulting from decimation over a spin which mediates between two sites is found via $\lambda(K_{\text{eff}}) = \lambda(K)\lambda(K_1)$. At low temperature, this implies that $K_{\text{eff}}^{-1} = K^{-1} + K_1^{-1}$. Thus, inverse coupling constants in parallel or in series add up exactly in the same way as resistors (see also Stinchcombe 1978). Indeed, for figure 2(a) we immediately find that $r' = 8/3r$, i.e. $\zeta = \ln(8/3)/\ln 3$.

The general result for models with continuous spin symmetry is thus that their exponent y is independent of n (for $n > 1$), and is equal to ζ . Note that $y_{n=1} < y_{n>1} < D$. The magnetic exponent x remains as in the Ising case, i.e. $x = D$. Since the magnetisation is discontinuous at $T = 0$, we again expect $\beta = 0$, i.e.

$$M(t, h) = b^0 m(b^y t, b^D h). \quad (4.8)$$

Integration with respect to h now yields the generalisation of equation (3.11), i.e.

$$F(t, h) = b^{-D} F(b^y t, b^D h). \quad (4.9)$$

These results are consistent with the statement that asymptotically the number of sites scales in the same way as the number of bonds.

The other critical exponents may now be obtained directly from equation (4.9). Again, one should be careful in using hyperscaling near $T = 0$ (Baker and Bonner 1975).

The simplest way to perform calculations on lattices containing interactions that are not part of the geometrical structure (e.g. the dotted lines in figure 2(g)) is the following. We perform the first renormalisation group iteration, in which we regard the lattice as if it was composed of all the interactions, including the 'dotted links'. After this step we obtain a renormalised branching curve, for which all the interactions are part of the lattice. Then we proceed as in § 4, using relations like (4.1).

5. Summary and discussion

In the present paper we studied a family of quasi-linear self-similar lattices, called Koch curves. These have minimum order of ramification equal to 1 (if they contain dangling ends, cf figure 2(h)), or 2. We distinguished between branching and non-branching Koch curves, the latter being homogeneously ramified. The finiteness of the order of ramification enabled us to perform exact calculations on these lattices, and was also responsible for the vanishing of the critical temperature for spins put on these lattices.

We showed that the fractal dimensionality D replaces the Euclidean dimensionality d in various scaling relations. This is true for any self-similar lattice (provided that the thermodynamic limit exists). Similar relations were recently derived for the percolation problem (Gefen *et al* 1982). For non-branching Koch curves $\zeta = y = x = D$ for both $n = 1$ and $n > 1$. However, for the branching Koch curves we find that $y_1 < y_{n>1} = \tilde{\zeta} < D$ and that the exponents ζ and y_1 do not depend only on D . In

addition, they depend on geometrical details of the orders of ramification at the various sites.

We also considered the critical behaviour due to magnetic field, and found that the exponent x is indeed equal to the fractal dimensionality (D) of the system, as expected for a discontinuity fixed point (Nienhuis and Nauenberg 1975).

We have verified that the recursion relations for $n > 1$ systems near $T = 0$ are equivalent to the problem of resistor networks on the corresponding lattices. The fact that these two problems are related was discussed by José *et al* (1977), Stinchcombe (1978).

We compared three quantities that scale under the renormalisation group iterations. These are the number of sites ($\sim b^D$), the effective resistance $R(\sim b^\xi)$ and the one-channel links $N'(\sim b^{\nu_l})$. These quantities define three effective lengths in the problem, that are related, respectively, to the total mass, the correlation length of $n > 1$ systems (or the resistance problem) and the Ising correlation length. The latter is also relevant to other magnetic systems with a discrete symmetry of the spin (e.g. Potts models, percolation, etc), whose analysis is similar to that of Ising systems. The existence of three diverging lengths in the percolation problem was recognised by Lubensky (1976) and Coniglio (1981), who used a quasi-linear picture to describe the backbone at percolation. However, the relevance of (possibly randomised) Koch curves to the percolation problem is not at all obvious (Gefen *et al* 1981). They probably become physically relevant for systems at high dimensionalities.

Again we emphasise that the fractals' being exactly solvable makes them a convenient tool to study generalisations of dimensionality and the effects of various geometrical and topological factors on critical behaviour.

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